THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH2010C/D Advanced Calculus 2019-2020

Solution to Assignment 1

1. In $\triangle ABC$, $\overrightarrow{AB} = 4\mathbf{i} + 4\mathbf{j}$, $\overrightarrow{AC} = -12\mathbf{i} + 8\mathbf{j}$ and points P, Q lie on BC such that $BP : PQ : QC = 1 : 2 : 1$.

   Find $\angle PAQ$.

   Ans: $\overrightarrow{AP} = \frac{3}{4}\overrightarrow{AB} + \frac{1}{4}\overrightarrow{AC} = \frac{3}{4}(4\mathbf{i} + 4\mathbf{j}) + \frac{1}{4}(-12\mathbf{i} + 8\mathbf{j}) = 5\mathbf{j}$.

   Similarly, $\overrightarrow{AQ} = \frac{1}{4}\overrightarrow{AB} + \frac{3}{4}\overrightarrow{AC} = \frac{1}{4}(4\mathbf{i} + 4\mathbf{j}) + \frac{3}{4}(-12\mathbf{i} + 8\mathbf{j}) = -8\mathbf{i} + 7\mathbf{j}$.

   Therefore, $\cos\angle PAQ = \frac{\overrightarrow{AP} \cdot \overrightarrow{AQ}}{|\overrightarrow{AP}||\overrightarrow{AQ}|} = \frac{35}{5\sqrt{113}}$ and $\angle PAQ = \cos^{-1}\left(\frac{7}{\sqrt{113}}\right)$.

2. Let $A = (4, 3, 6)$, $B = (-2, 0, 8)$ and $C = (1, 5, 0)$ be points in $\mathbb{R}^3$.

   Show that $\triangle ABC$ is a right-angled triangle.

   Ans: $\overrightarrow{AB} = (-2, 0, 8) - (4, 3, 6) = (-6, -3, 2)$ and $\overrightarrow{AC} = (1, 5, 0) - (4, 3, 6) = (-3, 2, -6)$.

   Then, $\overrightarrow{AB} \cdot \overrightarrow{AC} = (-6)(-3) + (-3)(2) + (2)(-6) = 0$ and so $AB \perp AC$.

   Therefore, $\triangle ABC$ is a right-angled triangle.

3. Suppose that $\mathbf{m}, \mathbf{n} \in \mathbb{R}^n$, where $|\mathbf{m}| = 2$, $|\mathbf{n}| = 1$ and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $\frac{2\pi}{3}$.

   If $\mathbf{p} = 3\mathbf{m} + 4\mathbf{n}$ and $\mathbf{q} = 2\mathbf{m} - \mathbf{n}$, find

   (a) $\mathbf{m} \cdot \mathbf{n}$,

   (b) $|\mathbf{p}|$ and $|\mathbf{q}|$,

   (c) the area of the parallelogram spanned by $\mathbf{p}$ and $\mathbf{q}$.

   Ans:

   (a) $\mathbf{m} \cdot \mathbf{n} = |\mathbf{m}||\mathbf{n}| \cos(\frac{2\pi}{3}) = -1$

   (b) $|\mathbf{p}|^2 = \mathbf{p} \cdot \mathbf{p} = (3\mathbf{m} + 4\mathbf{n}) \cdot (3\mathbf{m} + 4\mathbf{n}) = 9|\mathbf{m}|^2 + 24\mathbf{m} \cdot \mathbf{n} + 16|\mathbf{n}|^2 = 28$. Therefore, $|\mathbf{p}| = 2\sqrt{7}$.
   Similarly, $|\mathbf{q}|^2 = \mathbf{q} \cdot \mathbf{q} = (2\mathbf{m} - \mathbf{n}) \cdot (2\mathbf{m} - \mathbf{n}) = 4|\mathbf{m}|^2 - 4\mathbf{m} \cdot \mathbf{n} + |\mathbf{n}|^2 = 21$. Therefore, $|\mathbf{q}| = \sqrt{21}$.

   (c) We have $\mathbf{p} \cdot \mathbf{q} = (3\mathbf{m} + 4\mathbf{n}) \cdot (2\mathbf{m} - \mathbf{n}) = 15$.
   Let the angle between $\mathbf{p}$ and $\mathbf{q}$ be θ. Then $\cos\theta = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{15}{14\sqrt{3}}$. Therefore, $\sin\theta = \frac{11}{14}$.
   The area of the parallelogram spanned by $\mathbf{p}$ and $\mathbf{q}$ is $|\mathbf{p}||\mathbf{q}| \sin\theta = 11\sqrt{3}$.

4. Suppose that A, B and C are points on $\mathbb{R}^2$ such that $OABC$ is a kite with $OA = OC$ and $AB = CB$. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

   (a) Express $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

   (b) By considering $AB = CB$, show that $\mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{c}$.

   (c) Hence, show that $OB \perp AC$.

   Ans:

   (a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{CB} = \mathbf{b} - \mathbf{c}$

(b) Since $AB = CB$, we have

$$\begin{aligned} |\mathbf{b}-\mathbf{a}|^2 &= |\mathbf{b}-\mathbf{c}|^2 \\ (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) &= (\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c}) \\ |\mathbf{b}|^2 - 2\mathbf{b}\cdot\mathbf{a} + |\mathbf{a}|^2 &= |\mathbf{b}|^2 - 2\mathbf{b}\cdot\mathbf{c} + |\mathbf{c}|^2 \\ \mathbf{b}\cdot\mathbf{a} &= \mathbf{b}\cdot\mathbf{c} \end{aligned}$$

Note that $OA = OC$, and so $|\mathbf{a}| = |\mathbf{c}|$.

(c) Form (b), we have $\mathbf{b}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{c}$ and so $\mathbf{b}\cdot(\mathbf{c}-\mathbf{a}) = 0$, i.e. $\overrightarrow{OB}\cdot\overrightarrow{AC} = 0$.

Therefore, $OB \perp AC$.

5. Let $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OC} = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(b) Find the volume of tetrahedron $OABC$.

(Hint: Its volume equals to $\frac{1}{6}\times$volume of parallelotope spanned by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.)

(c) By (a) and (b), find the distance from O to $\triangle ABC$.

Ans:

(a) Firstly, we have $\overrightarrow{AB} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\overrightarrow{AC} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Then,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 4 & 1 & -1 \end{vmatrix} = -\mathbf{i} + 2\mathbf{k}.$$

(b) $(\overrightarrow{OA} \times \overrightarrow{OB})\cdot\overrightarrow{OC} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 5 & 1 & 3 \end{vmatrix} = 1.$

Therefore, the volume of tetrahedron $OABC = \frac{1}{6} \times |(\overrightarrow{OA} \times \overrightarrow{OB})\cdot\overrightarrow{OC}| = \frac{1}{6}$.

(c) From (a), the area of $\triangle ABC = \frac{1}{2} \times |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{\sqrt{5}}{2}$.

Let h be the distance from O to $\triangle ABC$.

Note that h is just the height of the tetrahedron $OABC$ with base $\triangle ABC$.

Then, $\frac{1}{3} \times \frac{\sqrt{5}}{2} \times h = \frac{1}{6}$ and so $h = \frac{1}{\sqrt{5}}$.

6. Given $A = (3, -1, 3)$, $B = (0, 7, -2)$ and $C = (-9, 3, -3)$ be three points in $\mathbb{R}^3$.

(a) Find the coordinates of a point D if AC, BD are perpendicular and AD, BC are parallel.

(b) i. Find $\angle DCB$.

ii. Show that A, B, C, D are coplanar (i.e. lying on a same plane) and find the equation of the plane which contains them.

iii. Show that $ABCD$ is a square and find the area of it.

(c) $VABCD$ is a pyramid with base $ABCD$. If $V = (12, -14, -12)$,

i. find the volume of the pyramid;

ii. find the angle between the plane VAB and the base.

Ans:

(a) Note that $\overrightarrow{AC} = -12\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$, $\overrightarrow{BD} = \overrightarrow{OD} - (7\mathbf{j} - 2\mathbf{k})$, $\overrightarrow{AD} = \overrightarrow{OD} - (3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\overrightarrow{BC} = -9\mathbf{i} - 4\mathbf{j} - \mathbf{k}$. Since AD and BC are parallel, $\overrightarrow{AD} = \lambda\overrightarrow{BC}$ for some $\lambda \in \mathbb{R}$. Then,

$$\overrightarrow{OD} = (3\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + \lambda(-9\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = (3 - 9\lambda)\mathbf{i} - (1 + 4\lambda)\mathbf{j} + (3 - \lambda)\mathbf{k}.$$

Since AC and BD are perpendicular, $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$. Then,

$$\begin{aligned} (-12\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}) \cdot \overrightarrow{OD} - 40 &= 0 \\ -12(3 - 9\lambda) - 4(1 + 4\lambda) - 6(3 - \lambda) - 40 &= 0 \\ \lambda &= 1 \end{aligned}$$

Therefore, $\overrightarrow{OD} = -6\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$, i.e. $D = (-6, -5, 2)$.

(b) i. $\angle DCB = \cos^{-1}\left(\dfrac{\overrightarrow{CD} \cdot \overrightarrow{CB}}{|\overrightarrow{CD}||\overrightarrow{CB}|}\right) = \cos^{-1}(0) = \dfrac{\pi}{2}$.

ii. Direct computation shows that $\overrightarrow{CA} \cdot (\overrightarrow{CD} \times \overrightarrow{CB}) = 0$ which implies A, B, C, D are coplanar. Also, $\overrightarrow{CD} \times \overrightarrow{CB}$ gives a normal of the plane containing A, B, C, D. The equation of the plane is $2x - 3y - 6z = -9$.

iii. Note that $\overrightarrow{AB} = \overrightarrow{DC} = -3\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}$ and $\overrightarrow{AD} = \overrightarrow{BC} = -9\mathbf{i} - 4\mathbf{j} - \mathbf{k}$. Therefore, $|\overrightarrow{AB}| = |\overrightarrow{DC}| = |\overrightarrow{AD}| = |\overrightarrow{BC}| = 7\sqrt{2}$. Furthermore, $\overrightarrow{AB} \cdot \overrightarrow{AD} = 0$ which shows that $\angle BAD = \dfrac{\pi}{2}$. Therefore, $ABCD$ is a square with area $= (7\sqrt{2})^2 = 98$.

(c) i. Let $\hat{n}$ be the unit vector of $\overrightarrow{CD} \times \overrightarrow{CB}$. Then, $\hat{n} = \dfrac{1}{7}(-2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$. Then, the height of the pyramid is $|\overrightarrow{BV} \cdot \hat{n}| = 21$. Therefore, the volume of the pyramid is $\dfrac{1}{3} \times 98 \times 21 = 686$.

ii. Let $\hat{m} = \dfrac{\overrightarrow{BV} \times \overrightarrow{BA}}{|\overrightarrow{BV} \times \overrightarrow{BA}|} = -\dfrac{1}{7\sqrt{886}}(185\mathbf{i} + 90\mathbf{j} + 33\mathbf{k})$. The angle between the plane VAB and the base $ABCD$ = the angle between $\hat{m}$ and $\hat{n} = \cos^{-1}(-\sqrt{\frac{2}{443}})$

7. Suppose that $L_1 : x+1 = \frac{y-2}{-2} = \frac{z+3}{2}$ and $L_2 : \frac{x-1}{-1} = \frac{y+2}{2} = \frac{z-6}{3}$ are two straight lines.

   (a) Show that L_1 and L_2 intersect each other at one point and find the point of intersection.

   (b) Find the acute angle between L_1 and L_2.

   (c) Find the equation of plane containing L_1 and L_2.

   Ans:

   (a) Rewrite the equations of L_1 and L_2 in parametric forms:

   $$\begin{aligned} L_1 : \quad & x = -1+s, y = 2-2s, z = -3+2s \\ L_2 : \quad & x = 1-t, y = -2+2t, z = 6+3t \end{aligned}$$

   where $s, t \in \mathbb{R}$.

   By setting $-1+s = 1-t$, $2-2s = -2+2s$ and $-3+2s = -6+3t$, we have the solution $s = 3$ and $t = -1$.

   Therefore, L_1 and L_2 intersects at $(2, -4, 3)$.

   (b) $\mathbf{d}_1 = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{d}_2 = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ are direction vectors of L_1 and L_2 respectively.

   Therefore, the angle between L_1 and $L_2 = \cos^{-1}\left(\frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}\right) = \cos^{-1}\left(\frac{1}{3\sqrt{14}}\right)$.

   (c) $\mathbf{d}_1 \times \mathbf{d}_2 = -10\mathbf{i} - 5\mathbf{j}$ is a normal of the required plane.

   Since $(2, -4, 3)$ is a point lying on the required plane, the required equation is $2x + y = 0$.

8. Let $\Pi_1 : x - 2y + 2z = 0$ and $\Pi_2 : 3x + y + 2z = 4$ be two planes and let $P(1, 2, -1)$ be a point in $\mathbb{R}^3$.

   (a) Find the angle between Π_1 and Π_2.

   (b) Find the equation of the line passing through the point P which is parallel to the intersection line of the planes Π_1 and Π_2.

   Ans:

   (a) Note that $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{n}_2 = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ are normals of Π_1 and Π_2 respectively.

   The angle between Π_1 and Π_2 = The angle between $\mathbf{n}_1$ and $\mathbf{n}_2 = \cos^{-1}(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}) = \cos^{-1}(\frac{5}{3\sqrt{14}})$.

   (b) Note that

   $$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 3 & 1 & 2 \end{vmatrix} = -6\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$$

   gives a direction vector of the intersection line of Π_1 and Π_2, and hence gives a direction vector of the required line.

   The required equation: $\frac{x-1}{-6} = \frac{y-2}{4} = \frac{z+1}{7}$.

9. Let $A = (1, 1, 0)$, $B = (0, 1, 1)$ and $C = (1, -1, 1)$ be three points in $\mathbb{R}^3$ and let Π be the plane containing A, B and C.

   (a) Find the equation of the plane Π.

   (b) Suppose that

   $$L : \frac{x-1}{5} = \frac{y-1}{6} = z$$

   is a straight line passing through the point A and L' is the projection of L on Π.

   Find the equation of L'.

Ans:

(a) $\overrightarrow{AB} = -\mathbf{i} + \mathbf{k}$ and $\overrightarrow{AC} = -2\mathbf{j} + \mathbf{k}$. Then,

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

gives a normal vector of the plane Π.

Let the equation of Π be $2x + y + 2z + D = 0$.

Note that $A = (1, 1, 0)$ is lying on Π, so $3 + D = 0$ and $D = -3$.

The equation of Π is $2x + y + 2z - 3 = 0$.

(b) $\mathbf{a} = 5\mathbf{i} + 6\mathbf{j} + \mathbf{k}$ is a direction vector of L. Then,

$$\mathbf{a} - \mathrm{proj}_{\mathbf{n}}(\mathbf{a}) = (5\mathbf{i} + 6\mathbf{j} + \mathbf{k}) - \frac{(5\mathbf{i} + 6\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{|2\mathbf{i} + \mathbf{j} + 2\mathbf{k}|^2}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

gives a direction vector of L'. Therefore, the equation of L' is

$$L' : x - 1 = \frac{y-1}{4} = -\frac{z}{3}.$$

10. (a) Let Π be a plane in $\mathbb{R}^3$ given by the equation $Ax + By + Cz + D = 0$ and let $P(x_0, y_0, z_0)$ be a fixed point. Show that the perpendicular distance between Π and P is $\left|\dfrac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$.

(b) Let $\Pi_1 : 2x - 2y + z - 4 = 0$ and $\Pi_2 : x + 2y - 2z = 0$ be two planes in $\mathbb{R}^3$.

Find the equation of plane(s) passing through the intersection lines of plane bisecting the planes Π_1 and Π_2.

(Hint: Suppose that $\mathbf{p}$ is a point lying on the required plane, then the distance between $\mathbf{p}$ and Π_1 equals to the distance between $\mathbf{p}$ and Π_2. Draw a picture to see why there are two such planes.)

Ans:

(a) Note that $\vec{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to Π. Let $Q = (x_1, y_1, z_1)$ be a fixed point on Π.

Since Q lies on Π, we have $Ax_1 + By_1 + Cz_1 = -D$.

Let θ be the angle between $\vec{n}$ and $\overrightarrow{PQ}$. Then, the perpendicular distance between Π and P

$$= \left| |\overrightarrow{PQ}| \cos\theta \right| = \left| \frac{|\overrightarrow{PQ}||\vec{n}| \cos\theta}{|\vec{n}|} \right| = \left| \frac{\overrightarrow{PQ} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)}{\sqrt{A^2 + B^2 + C^2}} \right| = \left| \frac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

(Note: $|-(Ax_0 + By_0 + Cz_0 + D)| = |Ax_0 + By_0 + Cz_0 + D|$.)

(b) Let $P = (x, y, z)$ be a point on the required plane.

Then, the distance between P and Π_1 equals to the distance between P and Π_2.

$$\begin{aligned} \left| \frac{2x - 2y + z - 4}{\sqrt{2^2 + (-2)^2 + 1^2}} \right| &= \left| \frac{x + 2y - 2z}{\sqrt{1^2 + 2^2 + (-2)^2}} \right| \\ 2x - 2y + z - 4 &= \pm(x + 2y - 2z) \end{aligned}$$

$x - 4y + 3z - 4 = 0$ and $3x - z - 4 = 0$ are two possible planes passing through the intersection lines of plane bisecting the planes Π_1 and Π_2.